# Math 2373: Linear Algebra and Differential Equations

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## Contents



## 1 Eigenvalues and Eigenvectors

It is a fundamental concept in linear algebra to study a linear operation $\mathcal{L}$, for example the multiplication of a vector by a matrix, by searching for those special vectors such that the input is proportional to the output:

$$\mathcal{L}[\vec{v}] = \lambda \vec{v},$$

for some real or complex number $\lambda$. One can for example relate a resonance, i.e. a steady-state periodic response (the output) of a system to a periodic forcing (the input) to an eigenvalue problem.

### 1.1 *Presentation and examples*

To be more precise, we let $A$ be a matrix of dimensions $n \times n$. Then $(\lambda, \vec{v})$ is an eigen-pair with $\lambda$ the *eigenvalue* and $\vec{v}$ the *eigenvector* when $\vec{v}$ is nonzero and

(1) $$A\vec{v} = \lambda \vec{v}.$$

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**Characteristic equation** We can characterize an *eigenvalue* by observing that the linear system:

$$(A - \lambda I_n)\vec{v} = \vec{0}$$

where $I_n$ is the identity matrix, has a nonzero solution if and only if it is underdetermined, that is if the determinant of $A - \lambda I_n$ is zero! This leads to the *characteristic equation* for eigenvalues:

$$\det(A - \lambda I_n) = 0. \tag{2}$$

**Eigenspaces** When $\lambda$ is an eigenvalue, i.e. a solution of the (polynomial) equation (2) above, then the homogeneous underdetermined system

$$(A - \lambda I_n)\vec{v} = \vec{0} \tag{3}$$

has a space of solutions, called the *eigenspace* of $\lambda$. Any nonzero element of this eigenspace is called an *eigenvector* associated with the *eigenvalue* $\lambda$.

*Practical steps to the determination of eigenvalues and eigenvectors.*

**Step 1** Assemble the characteristic equation by computing the determinant:

$$\det(A - \lambda I_n) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_{n-1,n-1}-\lambda & a_{n-1,n} \\ a_{n1} & \cdots & \cdots & a_{n,n-1} & a_{nn}-\lambda \end{vmatrix}$$

Note that one always obtains a polynomial of order $n$ as a result.

**Step 2** Obtain the eigenvalues $(\lambda_1, \ldots, \lambda_k$ with $k \leq n$ by solving the polynomial equation,

$$\det(A - \lambda I_n) = 0.$$

**Step 3** For each eigenvalue $\lambda_i$, solve by the Gauß-Jordan method the underdetermined system

$$(A - \lambda I_n)\vec{v} = \vec{0},$$

by obtaining the RREF form of $A - \lambda I_n$.

In practice, this is only doable by hand for $n = 2$.

Examples Let us start by looking at an easy $2 \times 2$ example:

$$A = \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix}.$$

As our first step, we introduce the variable $\lambda$ and we compute the determinant

$$\det(A - \lambda I_2) = \begin{vmatrix} 5-\lambda & 7 \\ -2 & -4-\lambda \end{vmatrix} = (5-\lambda)(-4-\lambda) - 7\cdot(-2) = \lambda^2 - \lambda - 6.$$

Next, we solve the characteristic equation for the eigenvalues,

$$\lambda^2 - \lambda - 6 = 0.$$

This is a second order polynomial equation, so we compute the discriminant, $\Delta = 1 + 24 = 25$, which is positive.

The eigenvalues of $A$ are:

$$\lambda_1 = \frac{1-5}{2} = -2 \qquad \text{and} \qquad \lambda_2 = \frac{1+5}{2} = 3.$$

Finally, we find the eigenspaces by the Gauß-Jordan method. For the first eigenvalue $\lambda_1 = -2$, we solve the system

$$\begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + 2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \vec{0}.$$

We write this system in augmented matrix form and we compute the RREF,

$$\left[\begin{array}{cc|c} 7 & 7 & 0 \\ -2 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus the eigenspace is the space of solutions of the equation $x + y = 0$ (first line of the RREF), and we can take $y$ as a free variable.

The eigenspace of $A$ for the eigenvalue $\lambda_1 = -2$ is deduced as:

$$\vec{v} = s \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \text{with } s \in \mathbb{R}.$$

In particular, $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = -2$.

We then repeat this analysis for $\lambda_2 = 3$: we solve the system

$$\begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - 3 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ -2 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \vec{0}.$$

We write this system in augmented matrix form and we compute the RREF,

$$\left[\begin{array}{cc|c} 2 & 7 & 0 \\ -2 & -7 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 7/2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus the eigenspace is the space of solutions of the equation $x + 7/2y = 0$ (first line of the RREF), and we take $y$ as a free variable:

The eigenspace of $A$ for the eigenvalue $\lambda_2 = 3$ is

$$\vec{v} = s \begin{bmatrix} -7/2 \\ 1 \end{bmatrix} \qquad \text{with } s \in \mathbb{R}.$$

In particular, $\vec{v}_2 = \begin{bmatrix} -7/2 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_2 = -2$. ✓

For our next example, we study the following $3 \times 3$ matrix:

$$B = \begin{bmatrix} 0 & 2 & 2 \\ -1 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}.$$

The first step is to introduce the variable $\lambda$ and compute the eigenvalue characteristic equation:

$$\det(B - \lambda I_3) = \begin{vmatrix} -\lambda & 2 & 2 \\ -1 & -\lambda & -1 \\ 1 & 2 & 3-\lambda \end{vmatrix} = 0.$$

By developing along the first column, we have

$$\begin{aligned} \begin{vmatrix} -\lambda & 2 & 2 \\ -1 & -\lambda & -1 \\ 1 & 2 & 3-\lambda \end{vmatrix} &= +(-\lambda) \begin{vmatrix} (-\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} - (-1) \begin{vmatrix} 2 & 2 \\ 2 & 3-\lambda \end{vmatrix} + (1) \begin{vmatrix} 2 & 2 \\ -\lambda & -1 \end{vmatrix} \\ &= -\lambda(\lambda(\lambda-3)+2) + (2(3-\lambda)-4) + (-2+2\lambda) \\ &= -\lambda(\lambda^2 - 3\lambda + 2) + (2 - 2\lambda) + (-2 + 2\lambda) \\ &= -\lambda(\lambda-1)(\lambda-2). \end{aligned}$$

The three eigenvalues of $B$ are the roots of the expression above, i.e.

$$\lambda_1 = 0, \qquad \lambda_2 = 1, \qquad \lambda_3 = 2.$$

As our next step we look for the three eigenspaces associated to each eigenvalue of $B$. First, for $\lambda_1 = 0$ we look for solutions of the underdetermined homogeneous system,

$$B\vec{v} = \vec{0},$$

by computing the RREF form of the augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 0 \\ -1 & 0 & -1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since only the first and second columns are pivot columns, the third variable is free:

The eigenspace of $B$ for the eigenvalue $\lambda_1 = 0$ is

$$\vec{v} = s \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \qquad \text{for } s \in \mathbb{R}.$$

In particular, $\vec{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector associated with $\lambda_1 = 0$.

Similarly, for $\lambda_2 = 1$ we look for solutions of the underdetermined homogeneous system,

$$B\vec{v} = \vec{v},$$

by computing the RREF form of the augmented matrix

$$\left[\begin{array}{ccc|c} -1 & 2 & 2 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & 2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Again only the first and second columns are pivot columns, and the third variable is free:

The eigenspace of $B$ for the eigenvalue $\lambda_2 = 1$ is

$$\vec{v} = s \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad \text{for } s \in \mathbb{R}.$$

In particular, $\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector associated with $\lambda_2 = 1$.

Finally, for $\lambda_3 = 2$ we look for solutions of the underdetermined homogeneous system,

$$B\vec{v} = 2\vec{v},$$

by computing the RREF form of the augmented matrix

$$\left[\begin{array}{ccc|c} -2 & 2 & 2 & 0 \\ -1 & -2 & -1 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1/3 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Again only the first and second columns are pivot columns, and the third variable is free:

The eigenspace of $B$ for the eigenvalue $\lambda_3 = 2$ is

$$\vec{v} = s \begin{bmatrix} 1/3 \\ -2/3 \\ 1 \end{bmatrix} \qquad \text{for } s \in \mathbb{R}.$$

In particular, $\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ is an eigenvector associated with $\lambda_3 = 1$.

Complex eigenvalues Sometimes, the characteristic polynomial does not have (only) real roots and the eigenvalues can only be computed by using complex numbers. An example is the simple matrix,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Indeed, the corresponding characteristic polynomial is

$$\det(A - \lambda I_2) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1.$$

Clearly, the characteristic equation $\det(A - \lambda I_2) = 0$ has two complex roots $\lambda_1 = -i$ and $\lambda_2 = +i$, which are the complex eigenvalues of the matrix. We may then proceed as before, with the only difference being the use of complex arithmetic, in solving the linear systems

$$A\vec{v} = \pm i\vec{v}.$$

For example, the eigenspace associated with the eigenvalue $\lambda_1 = -i$ is obtained by using Gauß-Jordan reduction:

$$\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow \overline{(i)} \cdot R_1} \left[\begin{array}{cc|c} 1 & -i & 0 \\ -1 & i & 0 \end{array}\right] \xrightarrow[R_2 \leftarrow R_1 + R_2]{} \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right],$$

where we will stress the use of the complex conjugate of the pivot entry as a row multiplier, here $-i = \overline{(i)}$, used to transform the pivot into a real number. The second variable is then free:

The eigenspace of $A$ for the eigenvalue $\lambda_1 = -i$ is

$$\vec{v} = s \begin{bmatrix} i \\ 1 \end{bmatrix} \qquad \text{for } s \in \mathbb{C}.$$

In particular, $\vec{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ is an eigenvector associated with $\lambda_1 = -i$.

The second eigenspace associated with the eigenvalue $\lambda_2 = i$ is obtained also by using Gauß-Jordan reduction:

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow \overline{(-i)} \cdot R_1} \left[\begin{array}{cc|c} 1 & i & 0 \\ -1 & -i & 0 \end{array}\right] \xrightarrow[R_2 \leftarrow R_1 + R_2]{} \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right],$$

where we again use the complex conjugate of the pivot entry as a row multiplier, here $i = \overline{(-i)}$. As before, the second variable is free:

The eigenspace of $A$ for the eigenvalue $\lambda_2 = i$ is

$$\vec{v} = s \begin{bmatrix} -i \\ 1 \end{bmatrix} \qquad \text{for } s \in \mathbb{C}.$$

In particular, $\vec{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ is an eigenvector associated with $\lambda_2 = +i$.

## 1.2 *Repeated eigenvalues are trouble!*

As with second order differential equations, the case where the characteristic polynomial has repeated roots is special and should be treated with care. An example of bad behavior is the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Indeed, its characteristic polynomial is

$$\det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2,$$

which has a unique double root $\lambda = 1$, which is the only eigenvalue of the matrix. When we look at the associated eigenvectors, we proceed by Gaussian elimination,

$$\left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

This matrix has one pivot column, the second, and the first variable is free. The space of solutions of the corresponding homogeneous, underdetermined linear system is thus

$$\vec{v} = s \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

In particular, one cannot find two eigenvectors of the matrix which are *linearly independent!* We will discuss this problem again in our presentation of diagonalization; before moving on, we state a general dimension counting rule:

> Let $A$ be a $n \times n$ square matrix, with $m$ eigenvalues $\lambda_1, \ldots, \lambda_m$ $(m \leq n)$ which are the roots of the characteristic polynomial. Thus we can write
>
> $$\det (A - \lambda I_n) = (\lambda - \lambda_1)^{k_1} \cdots (\lambda - \lambda_m)^{k_m},$$
>
> where $k_1, \ldots, k_m$ are the *algebraic multiplicities* of the corresponding eigenvalues, with
>
> $$1 \leq k_i \leq n \qquad \text{and} \qquad k_1 + \cdots + k_m = n.$$
>
> When $k_i > 1$, the corresponding root $\lambda_i$ is *repeated* and we may have more than one corresponding independent eigenvector. In fact, if we call $p_i$ the number of independent eigenvectors associated with $\lambda_i$ (also called the *dimension* of the eigenspace of $A$ associated with $\lambda_i$), then
>
> $$p_i = n - (\text{Number of pivot columns in the RREF of } A - \lambda_i I_n).$$
>
> The $p_i$ are the *geometric multiplicities* of the eigenvalues, and they may not equal their algebraic multiplicities!

Note that we always have

$$1 \leq p_i \leq k_i.$$

In particular, when $k_i = 1$ then $p_i = 1$. However, when $k_i > 1$, *there may sometimes be strictly less free eigenvectors associated with $\lambda_i$ than the algebraic multiplicity ($p_i < k_i$)* as in the exemple above. In this case, the total number of free eigenvectors is less than $n$ and they do not form a basis of the space $\mathbb{R}^n$.

## 2 Diagonalization

The identification of the eigenvalues and eigenvectors of a matrix allows us to introduce an important *decomposition* of matrices: the diagonalization, i.e. a change of basis through which the matrix becomes *diagonal*. Geometrically, a matrix corresponds to a linear transformation of space. The diagonalization corresponds to an identification of the directions of space in which the transformation acts as a pure dilatation or contraction. These directions are then used to construct a new set of coordinates, in which the matrix corresponding to the transformation is a simple diagonal matrix.

## 2.1 *Presentation and Examples*

Change of basis To start with, let us consider a $2 \times 2$ matrix $A$ with two eigenpairs $(\lambda_1, \vec{v}_1)$, $(\lambda_2, \vec{v}_2)$, with distinct eigenvalues $\lambda_1 \neq \lambda_2$. A point of the plane $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is usually identified by its cartesian coordinates $(x_1, x_2)$:

$$X = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

However, this same point can also be identified by its coordinates $(a_1, a_2)$ in the basis of eigenvectors $(\vec{v}_1, \vec{v}_2)$:

$$X = a_1\vec{v}_1 + a_2\vec{v}_2.$$

To go from the cartesian coordinates $(x_1, x_2)$ to the new coordinates $(a_1, a_2)$, we form the matrix $P$ whose columns are the eigenvectors $\vec{v}_1$ and $\vec{v}_2$): we write schematically

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}.$$

Then,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}.$$

To go in the other direction, we use the inverse of the matrix $P$:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = P^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P^{-1}X.$$

Using this change of basis and the corresponding matrix $P$, we can compute

$$\begin{aligned} AX = A(a_1\vec{v}_1 + a_2\vec{v}_2) &= a_1 A\vec{v}_1 + a_2 A\vec{v}_2 \\ &= a_1\lambda_1\vec{v}_1 + a_2\lambda_2\vec{v}_2 = P \begin{bmatrix} \lambda_1 a_1 \\ \lambda_2 a_2 \end{bmatrix}. \end{aligned}$$

Now, let us introduce the diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

We continue our calculation:

$$\begin{aligned} \begin{bmatrix} \lambda_1 a_1 \\ \lambda_2 a_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= D \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \\ &= DP^{-1}X. \end{aligned}$$

To conclude, we have the following identity for any point $X$ in space:

$$AX = PDP^{-1}X,$$

and thus we have that $A = PDP^{-1}$.

The square matrix $A$ of order $n$ is diagonalizable if there is an invertible matrix $P$ of order $n$ and a diagonal matrix $D$ of order $n$ such that, in the original basis, i.e. applied to the vector of coordinates $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$:

$$A = PDP^{-1}.$$

In the basis of eigenvectors, i.e. applied to the vector $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$:

$$D = P^{-1}AP.$$

Finally, we have the mixed formulation

$$AP = PD.$$

All three identities can be used equivalently.

- It is not always possible to diagonalize a matrix!
- However, any matrix of order $n$ with $n$ *distinct* eigenvalues is diagonalizable.

Example: $2 \times 2$ matrix As a first example, we diagonalize the matrix $A = \begin{bmatrix} 3 & 1 \\ 5 & 3 \end{bmatrix}$. We proceed first by obtaining the eigenvalues and eigenvectors of $A$ as in the preceding section:

*Step 1:* determine the characteristic polynomial:

$$\det(A - \lambda I_2) = \begin{vmatrix} 3-\lambda & 1 \\ 5 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 5 = \lambda^2 - 6\lambda + 4.$$

*Step 2:* solve the characteristic equation for the eigenvalues:

$$\lambda^2 - 6\lambda + 4 = 0 \quad \rightsquigarrow \quad \Delta = 36 - 16 = 20 \quad \rightsquigarrow \quad \lambda_1 = 3 - \sqrt{5}, \quad \lambda_2 = 3 + \sqrt{5}.$$

*Step 3:* solve for two eigenvectors corresponding to each eigenvalue. For $\lambda_1 = 3 - \sqrt{5}$, we solve the underdetermined, homogeneous system,

$$\begin{bmatrix} 3 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (3 - \sqrt{5}) \begin{bmatrix} x \\ y \end{bmatrix}.$$

Applying the Gauß-Jordan method, we form the augmented matrix and find its RREF,

$$\left[\begin{array}{cc|c} \sqrt{5} & 1 & 0 \\ 5 & \sqrt{5} & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 1/\sqrt{5} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus the second variable $y$ is free and choosing its value as 1, we find a first eigenvector

$$\vec{v}_1 = \begin{bmatrix} -1/\sqrt{5} \\ 1 \end{bmatrix}.$$

In the same way, we find an eigenvector for $\lambda_2 = 3 + \sqrt{5}$ by solving the underdetermined, homogeneous system,

$$\begin{bmatrix} 3 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (3 + \sqrt{5}) \begin{bmatrix} x \\ y \end{bmatrix}.$$

Applying the Gauß-Jordan method, we form the augmented matrix and find its RREF,

$$\left[\begin{array}{cc|c} -\sqrt{5} & 1 & 0 \\ 5 & -\sqrt{5} & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -1/\sqrt{5} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus the second variable $y$ is free and choosing its value as 1, we find a second eigenvector

$$\vec{v}_2 = \begin{bmatrix} 1/\sqrt{5} \\ 1 \end{bmatrix}.$$

*Step 4:* assemble the matrices $P$ and $D$:

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} & 1/\sqrt{5} \\ 1 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 3 - \sqrt{5} & 0 \\ 0 & 3 + \sqrt{5} \end{bmatrix}.$$

*Step 5:* check that everything is correct:

$$AP = PD = \begin{bmatrix} \frac{\sqrt{5}-3}{\sqrt{5}} & \frac{\sqrt{5}+3}{\sqrt{5}} \\ 3 - \sqrt{5} & 3 + \sqrt{5} \end{bmatrix}. \qquad \checkmark$$

Example: $3 \times 3$ matrix Next we investigate the diagonalization of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}.$$

*Step 1:* Dompute the characteristic polynomial,

$$\det(A-\lambda I_3) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{vmatrix}.$$

We develop this determinant along the third column:

$$\begin{aligned} \det(A-\lambda I_3) &= +1\cdot \begin{vmatrix} 6 & -1-\lambda \\ -1 & -2 \end{vmatrix} + (-1-\lambda)\cdot \begin{vmatrix} 1-\lambda & 2 \\ 6 & -1-\lambda \end{vmatrix} \\ &= (-12-(1+\lambda)) + (1+\lambda)\left((1+\lambda)(1-\lambda)+12\right) \\ &= \lambda(-\lambda^2-\lambda+12) \\ &= -\lambda(\lambda-3)(\lambda+4). \end{aligned}$$

*Step 2:* Determine the eigenvalues,

$$\lambda_1 = -4, \quad \lambda_2 = 0, \quad \lambda_3 = 3.$$

*Step 3:* Determine the eigenvectors. For the first eigenvalue $\lambda_1 = -4$, we must solve the homogeneous system

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -4\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Using the augmented matrix format, we reduce to RREF form:

$$\left[\begin{array}{ccc|c} 5 & 2 & 1 & 0 \\ 6 & 3 & 0 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The first and second column are pivot columns, but the $z$ variable is free, and we fix its value as 1 to get a first eigenvector:

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

For the second eigenvalue $\lambda_2 = 0$, we proceed in the same way to solve

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Using the augmented matrix format, we reduce to RREF form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 6 & -1 & 0 & 0 \\ -1 & -2 & -1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 1/13 & 0 \\ 0 & 1 & 6/13 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The first and second column are pivot columns, but the $z$ variable is free, and we fix its value as $-13$ to get a second eigenvector:

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix}.$$

For the third eigenvalue $\lambda_3 = 0$, we proceed in the same way again:

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3 \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Using the augmented matrix format, we reduce to RREF form:

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 6 & -4 & 0 & 0 \\ -1 & -2 & -4 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The first and second column are pivot columns, but the $z$ variable is free, and we fix its value as $-2$ to get a third eigenvector:

$$\vec{v}_3 = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}.$$

*Step 4:* We assemble $P$ and $D$:

$$P = \begin{bmatrix} -1 & 1 & 2 \\ 2 & 6 & 3 \\ 1 & -13 & -2 \end{bmatrix}, \qquad D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

*Step 5:* We check our result. Here

$$AP = PD = \begin{bmatrix} 4 & 0 & 6 \\ -8 & 0 & 9 \\ -4 & 0 & -6 \end{bmatrix}. \qquad \checkmark$$

Example: $2 \times 2$ matrix with complex eigenvalues As our third example, we study the case of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ -2 & 4 \end{bmatrix}.$$

*Step 1:* characteristic polynomial.

$$\det(A - \lambda I_2) = (2-\lambda)(4-\lambda) + 2 = \lambda^2 - 6\lambda + 10.$$

*Step 2:* solve $\lambda^2 - 6\lambda + 10 = 0$ for the eigenvalues.

$$\Delta = 36 - 40 = -4 \quad \Longrightarrow \quad \lambda_1 = \frac{6-2i}{2} = 3 - i \quad \text{and} \quad \lambda_2 = 3 + i.$$

Thus the roots of the characteristic polynomial, i.e. the eigenvalues, have complex values. The eigenspace associated with the eigenvalue $\lambda_1 = 3 - i$ is obtained by using Gauß-Jordan reduction:

$$\left[\begin{array}{cc|c} -1+i & 1 & 0 \\ -2 & 1+i & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow \overline{(-1+i)} \cdot R_1} \left[\begin{array}{cc|c} 2 & -1-i & 0 \\ -2 & 1+i & 0 \end{array}\right] \xrightarrow[R_2 \leftarrow R_1 + R_2]{} \left[\begin{array}{cc|c} 1 & -\frac{1+i}{2} & 0 \\ 0 & 0 & 0 \end{array}\right],$$

where we will stress the use of the complex conjugate of the pivot entry as a row multiplier, here $-1 - i = \overline{(-1+i)}$, used to transform the pivot into a real number. The second variable is then free, and in particular

$$\vec{v}_1 = \begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix}$$

is an eigenvector associated with $\lambda_1 = 3 - i$.

The second eigenspace associated with the eigenvalue $\lambda_2 = 3 + i$ is obtained also by using Gauß-Jordan reduction:

$$\left[\begin{array}{cc|c} -1-i & 1 & 0 \\ -2 & 1-i & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow \overline{(-1-i)} \cdot R_1} \left[\begin{array}{cc|c} 2 & i-1 & 0 \\ -2 & 1-i & 0 \end{array}\right] \xrightarrow[R_2 \leftarrow R_1 + R_2]{} \left[\begin{array}{cc|c} 1 & \frac{i-1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right],$$

where we again use the complex conjugate of the pivot entry as a row multiplier, here $-1 + i = \overline{(-1-i)}$. As before, the second variable is free, and in particular

$$\vec{v}_2 = \begin{bmatrix} \frac{1-i}{2} \\ 1 \end{bmatrix} \text{ is an eigenvector associated with } \lambda_2 = 3 + i.$$

*Step 4:* Assemble $P$ and $D$:

$$P = \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ 1 & 1 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 3-i & 0 \\ 0 & 3+i \end{bmatrix}.$$

*Step 5:* Practical check,

$$AP = PD = \begin{bmatrix} 2+i & 2-i \\ 3-i & 3+i \end{bmatrix}. \qquad \checkmark$$

## 2.2 *Applications*

While the theory of eigenvalues and diagonalization has a tremendous range of applications in all of physics (in particular quantum mechanics is based on eigenvalues!), engineering, computer science, it is out of the scope of this note to discuss this - a quick Google search will give some examples (e.g. the Google PageRank algorithm).

From our limited experience here, we can already see some of the practical benefits of knowing the decomposition of the matrix in a diagonal form, $A = P^{-1}DP$, when it exists.

*Computing the determinant:* since the determinant of a product of matrices is the product of the determinants, we get

$$\det(A) = \det(P^{-1}DP) = \det(P^{-1})\det(D)\det(P) = \det(D)\det(P)/\det(P),$$

so that $\det(A) = \det(D) = \lambda_1 \cdots \lambda_n$. The determinant of a diagonal matrix is very easy to compute!

*Computing the inverse:* when none of the eigenvalues is zero, the matrix $A$ is invertible (since its determinant $\det(A) = \lambda_1 \cdots \lambda_n$ is nonzero!). In addition, we have the formula

$$A^{-1} = P^{-1}D^{-1}P \qquad \text{where } D^{-1} = \begin{bmatrix} 1/\lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & 1/\lambda_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1/\lambda_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & 1/\lambda_n \end{bmatrix}$$

Indeed, using the rules on the inverse of a product we know that

$$\left(P^{-1}DP\right)^{-1} = P^{-1}\left(P^{-1}D\right)^{-1} = P^{-1}D^{-1}\left(P^{-1}\right)^{-1} = P^{-1}D^{-1}P.$$

Since computing the inverse of a diagonal matrix is very easy, we see that the diagonalization of the matrix makes it quite easy to compute the inverse! In fact...

*Computing powers of the matrix:* It is often useful to know the powers of the matrix $A$, i.e. the repeated products

$$A^k = \overbrace{A \times A \times \cdots \times A}^{k \text{ times}} \quad \text{or} \quad A^{-k} = \overbrace{A^{-1} \times A^{-1} \times \cdots \times A^{-1}}^{k \text{ times}},$$

where $k$ is some positive integer. This is usually a costly computation, involving repeated matrix products! However, if we have successfully diagonalized the matrix $A$ then we can exploit the formula

$$A = PDP^{-1}.$$

Indeed, we have

$$A^k = \overbrace{PDP^{-1} \times PDP^{-1} \times \cdots \times PDP^{-1}}^{k \text{ times}} = P\overbrace{D \times D \times \cdots \times D}^{k \text{ times}}P^{-1} = PD^kP^{-1}$$

since all the intervening $P^{-1}P$ substitute for $I$, the identity matrix, and can be removed from the product. Then the key is that $D^k$ is very easy to compute! It is just

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2^k & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \lambda_{n-1}^k & 0 \\ 0 & \cdots & \cdots & 0 & \lambda_n^k \end{bmatrix}.$$

One can also read Prof. Mori's note on this subject, which is on my webpage,

`http://math.umn.edu/~pcazeaux/teaching.html`

or at the course webpage,

`http://www.math.umn.edu/~gwanders/Math2373/matrix_powers.pdf`.

### 2.3 *Troublemakers: repeated eigenvalues and diagonalizability*

Any square matrix of size $n$ with $n$ distinct eigenvalues can be diagonalized if we admit complex eigenvalues. However, this is not always the case when some eigenvalues have multiplicity greater than one, as seen in Section 1.2.

A case where it doesn't work. We can look at the same example as in Section 1.2:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

This matrix has the eigenvalue $\lambda = 1$ with algebraic multiplicity 2 and geometric multiplicity 1: the eigenspace of $A$ is the line

$$\vec{v} = s\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

With only one free eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we cannot form the $2 \times 2$ matrix $P$ we need for the diagonalization step! In fact, it is not possible to find such a matrix: if we suppose that

$$A = PDP^{-1},$$

then the characteristic polynomial of $D$ is the same as the one of $A$:

$$\det(A - \lambda I) = \det(PDP^{-1} - \lambda I) = \det(P(D - \lambda I)P^{-1}) = \det(D - \lambda I),$$

meaning that the only eigenvalue of $D$ is also 1. Since $D$ is diagonal, this means that

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus $D$ is the identity! As a consequence, $PDP^{-1} = PP^{-1} = I$, leading to the conclusion

$$A = I.$$

This is obviously false! Thus our original premise that $A$ can be written as $PDP^{-1}$ is wrong. *The matrix $A$ is not diagonalizable.*

A CASE WHERE IT WORKS. We now study the case of the matrix

$$B = \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Let us compute the eigenvalues of $B$: the characteristic polynomial is

$$\det(B - \lambda I) = \begin{vmatrix} -1-\lambda & -1 & 1 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (-1-\lambda)(-2-\lambda)(-1-\lambda)$$
$$= -(\lambda+1)^2(\lambda+2).$$

So we have a simple eigenvalue $\lambda_1 = -2$ with multiplicity 1, and a repeated eigenvalue $\lambda_2 = -1$ with algebraic multiplicity 2. Let us find the eigenvectors. For the eigenvalue $\lambda_1 = -2$, we solve the homogeneous system

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \xrightarrow{\text{RREF}} \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The RREF has two pivot columns, the first and third; so the second variable $y$ is free. The solutions of this system then are

$$\vec{v} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{with } s \in \mathbb{R}.$$

We can choose an eigenvector associated with $\lambda_1 = -2$ such as

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

For the eigenvalue $\lambda_2 = -1$, we solve the homogeneous system

$$\begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \xrightarrow{\text{RREF}} \quad \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the RREF has only one pivot column, the second, there is two free variables: $x$ and $z$. The set of solutions is

$$\vec{v} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \text{with } s, t \in \mathbb{R}.$$

and we obtain two independent eigenvectors for $\lambda_2 = -1$:

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

We thus have a set of three independent eigenvectors.

We assemble $P$ and $D$:

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$